Laminar Natural Convection to an Isothermal Flat Plate with a Spatially Varying Acceleration

ROBERT LEMLICH and J. STEVEN STEINKAMP
University of Cincinnati, Cincinnati, Ohio

The classical problem of laminar natural convection to a flat vertical isothermal plate has been the subject of a number of experimental and theoretical investigations. Among the latter are included a well-known analytical solution of the boundary layer equations (9), a recent numerical solution by finite difference approximations (4, 5), and a solution by the integral method (2). However these studies are for a spatially uniform acceleration such as terrestial gravity. Scant attention has been paid to the situation where the acceleration varies with location within the system. Accordingly the present study was conducted to consider such a situation, namely where the acceleration is parallel to the plate but is directly proportional to distance along the plate measured from the leading edge.

Physically this situation can be approximated (for a boundary layer which is not too thick) by a cool isothermal plate mounted along the diameter of a spinning satellite in orbit, as shown in Figure 1. Thus the leading "edge" of the plate is at the center of rotation, and the radial centrifugal acceleration is given as $\omega^2(x^2 + y^2)^{1/2}$. However in the vicinity of the plate the centrifugal acceleration reduces to $\omega^2 x$ parallel to the plate.

With Coriolis forces neglected, cooled denser fluid near the plate is thrown radially outward, to be replaced by warmer surrounding fluid, which for the problem at hand is presumed to be very large in bulk and of fixed temperature and properties. Were the plate warmer than its environmental fluid the situation would be different, partly because the leading edge would no longer be at the center of rotation and partly because a plug of heated fluid would accumulate at the center.

The physical system here also differs from that of a rotating plate in otherwise stationary surroundings. In the present case the environment rotates right along with the plate.

One of the present authors first attacked this problem by the integral method for a turbulent as well as a laminar boundary layer (7). For the laminar regime the boundary-layer thickness and the local heat transfer coefficient were found to be independent of location along the plate. (This contrasts with the usual case of uniform acceleration where both vary.) Equation (1) was derived as the overall steady state result for the laminar regime:

$$N_{Nu} = 0.546 \left[\frac{N_{Pr}}{N_{Pr} + 1.143} \right]^{1/4} (N_{Pr} \cdot N'_{Gr})^{1/4} \quad (1)$$

However by its very nature the integral method requires assumptions regarding the form of the velocity and temperature profiles. Thus its results are not entirely conclusive. The solution obtained was also limited to steady state. Therefore in the present study a more rigorous analysis of the laminar case is carried out which gives the unsteady state solution as well. The procedure involves a numerical solution of the finite difference equations which correspond to the differential conservation equations. Iteration in real time is employed, thus yielding first the unsteady state solution, followed by the steady state solution as the limit at long time.

Of course an alternative approach would be to first equate all temporal derivatives to zero. This procedure was used successfully without the usual boundary-layer assumptions for a more complicated system involving natural convection between horizontal coaxial cylinders under uniform acceleration (1). However such an approach is necessarily limited to steady state and so was not employed here.

It might be noted in passing that among natural convection problems in general, only a few finite difference solutions have appeared.

THEORY

In dimensionless form the equations for the conservation of energy, momentum, and mass respectively are as follows:

$$\frac{\partial \phi}{\partial \tau} + U \frac{\partial \phi}{\partial X} + V \frac{\partial \phi}{\partial Y} = \frac{1}{N_{Pr}} \left(\frac{\partial^2 \phi}{\partial Y^2} \right) \quad (2)$$

$$\frac{\partial U}{\partial \tau} + U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = \phi \cdot X + \left(\frac{\partial^2 U}{\partial Y^2} \right) \quad (3)$$

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \quad (4)$$

These of course are for negligible changes in fluid properties (except density in the buoyancy term), negligible

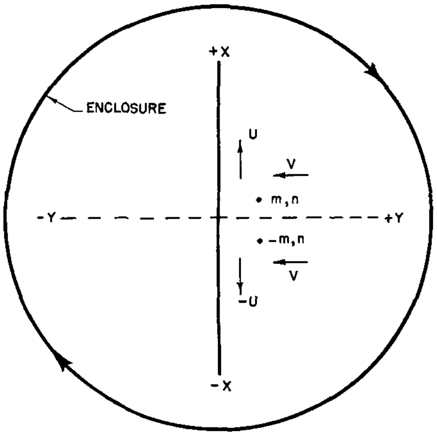

Fig. 1. Physical system. Rotation is about the Z axis.

J. Steven Steinkamp is with the Rocketdyne Division, North American Aviation, Canoga Park, California.

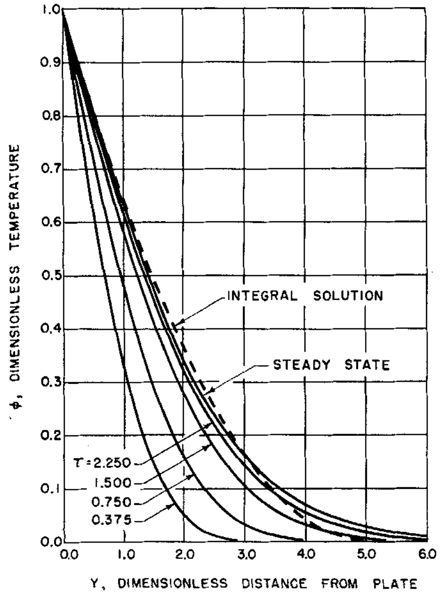

Fig. 2. Transient and steady dimensionless temperature profiles for a Prandtl number of 0.71.

viscous dissipation, and negligible changes in potential energy. Derivatives $(\partial^2\phi/\partial X^2)$ and $(\partial^2 U/\partial X^2)$ as well as the entire momentum balance for the Y direction are omitted for the usual classical reasons (9).

The differential equations thus resemble those for the classical case of uniform acceleration, except that g is replaced by $\omega^2 x$. The dimensionless groupings employed obviate the need for Grashof number as a parameter.

The boundary conditions are as follows:

$U = 0$	at $X = 0$
$U = V = 0$ and $\phi = 1$	at $Y = 0$
$U = \phi = 0$	at $Y = \infty$
$U = V = \phi = 0$	at $\tau = 0$

These are the same boundary conditions of those for uniform acceleration (3, 10), with one important exception. The condition of $\phi = 0$ at $X = 0$ does not apply here because the boundary-layer thickness is not zero at the leading edge. Instead use is made of symmetry around the edge (which is at the center of rotation). Thus ϕ, V, and U at location (X, Y) equal ϕ, V, and $-U$ respectively at location $(-X, Y)$.

Next differential Equations (2), (3), and (4) are replaced by corresponding explicit finite difference equations. These two-dimensional equations can then be simplified by showing that at time $\tau = \Delta\tau$ (that is the first step in time) ϕ and V are functions only of Y and τ, and U for a particular Y and τ is directly proportional to X. Then substituting the mathematical equivalents for these statements (into the aforementioned two-dimensional difference equations) yields similar statements for ϕ, V, and U at any subsequent time, including steady state. Details of this algebraic analysis have been placed on file (12). [The results are equivalent to those obtainable by first assuming the statements to be true and then substituting their mathematical equivalents directly into the differential Equations (2), (3), and (4).]

This independence of X for ϕ and V, and the proportionality of U to X, are in agreement with the steady state integral solution mentioned earlier.*

* The simplification for ϕ also means that for the present situation of spatially varying acceleration, uniform surface temperature implies uniform surface heat flux. This simple equivalence between the two boundary conditions is in marked contrast to the classical situation of uniform acceleration in which the two are dissimilar.

For computational purposes the problem is now reduced to one dimension. Equations (5), (6), and (7) are the resulting difference equations for $X = 1$:

$$(\phi_n' - \phi_n)/\Delta\tau + V_n\,(\phi_{n+1} - \phi_n)/\Delta Y = (\phi_{n+1} - 2\phi_n + \phi_{n-1})/N_{Pr}\,(\Delta Y)^2 \quad (5)$$

$$(U_n' - U_n)/\Delta\tau + U_n^2 + V_n\,(U_{n+1} - U_n)/\Delta Y = \phi_n' + (U_{n+1} - 2U_n + U_{n-1})/(\Delta Y)^2 \quad (6)$$

$$U_n + (V_n - V_{n-1})/\Delta Y = 0 \quad (7)$$

Integer n is defined by $Y = (n - 1)\,\Delta Y$. The choice of forward or backward differences here for the first-order terms involves stability considerations (4). Stability restrictions are as follows:

$$|V|/\Delta Y + 2/N_{Pr}\,(\Delta Y)^2 \leqq 1/\Delta\tau \quad (8)$$

$$U + |V|/\Delta Y + 2/(\Delta Y)^2 \leqq 1/\Delta\tau \quad (9)$$

DISCUSSION AND RESULTS

Computations were first carried out at a Prandtl number of 0.71. Fortran II was employed with an IBM-1620 computer. In addition, some calculations were deliberately carried out without the one-dimensional simplification discussed above. Results for such two-dimensional calculations agreed with those for one dimension as would be expected.

The increments employed in the computation for N_{Pr} of 0.71 were $\Delta Y = 0.22$ and $\Delta\tau = 0.015$. Figure 2 presents the resulting transient and steady dimensionless temperature profiles. These do not vary along the plate. Figure 3 presents the resulting transient and steady dimensionless velocity profiles. A rapid initial buildup in both profiles is evident.

In Figure 4 the numerical solution for transient $N_{Nu}(N'_{Gr})^{-1/4}$ is compared with the analytical solution for one-dimensional heat conduction (6) which should be nearly correct at small τ. The two solutions agree well with each other.

It is interesting to note the fairly close comparison between steady state values for $N_{Nu}(N'_{Gr})^{-1/4}$ determined here for nonuniform acceleration and those reported for $N_{Nu}(N_{Gr})^{-1/4}$ with uniform acceleration. For example at $N_{Pr} = 0.71$ the former is 0.374, while the latter (via interpolation) is 0.355 for uniform surface temperature (8) and about 0.42 for uniform surface heat flux (11). This would seem to imply that insofar as the Nusselt number is

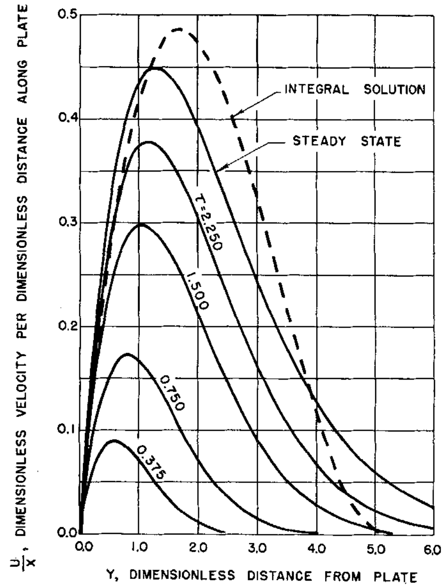

Fig. 3. Transient and steady dimensionless velocity profiles for a Prandtl number of 0.71.

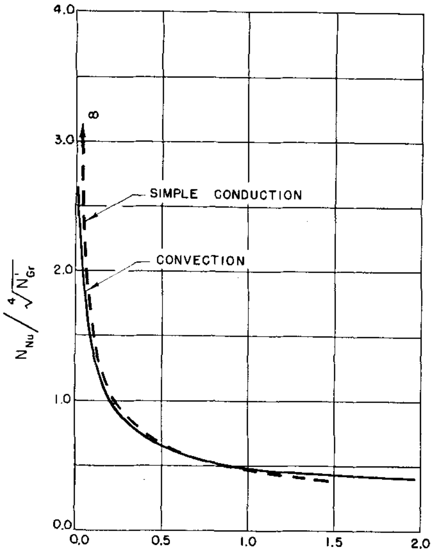

Fig. 4. Comparison of transient solutions at short times for a Prandtl number of 0.71.

concerned $\omega^2 x$ in the present system plays roughly the same physical role as g in the classical case of uniform acceleration.

The numerical computation was extended to $N_{Pr} = 0.01$ and $N_{Pr} = 10$. Figure 5 shows overall results at steady state for all N_{Pr}.

For comparison with the present analysis, Figures 3, 4, and 5 also show results of the integral method. Agreement is good. Values for $N_{Nu}(N'_{Gr})^{-1/4}$ calculated by the two different methods agree within 5% at $N_{Pr} = 0.71$.

The very good agreement with conduction theory at short times, coupled with the good agreement with the integral method at long times (steady state), support the validity of the results at intermediate times.

SUMMARY

By replacing the differential equations of conservation with corresponding finite difference equations, a solution was obtained by digital computation for laminar natural convection to a flat isothermal plate with a parallel acceleration proportional to distance along the plate measured from the leading edge. For step function transient conditions or steady conditions the local coefficient of heat transfer was shown to be independent of location along the plate. Comparison of results at steady state with those for a recent solution by the integral method yielded good agreement. Comparison with theory at short times also yielded good agreement.

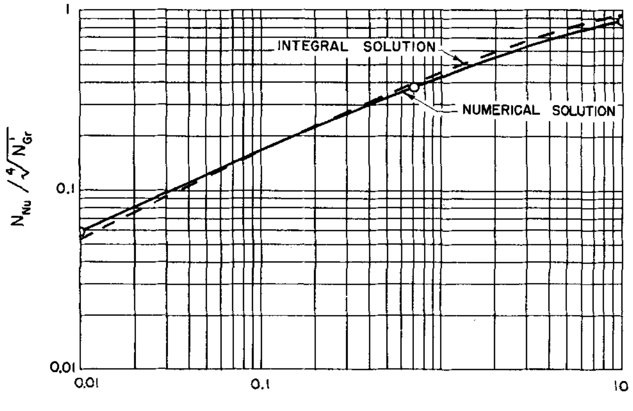

Fig. 5. Overall results at steady state for various Prandtl numbers.

ACKNOWLEDGMENT

This work was indirectly supported in part by National Science Foundation research grant number G19281 at the University of Cincinnati Computing Center.

NOTATION

c_p = heat capacity at constant pressure
g = uniform acceleration such as gravity
h = local heat transfer coefficient
k = thermal conductivity
N'_{Gr} = Grashof number, $\omega^2 x^4 \beta \Delta T/\nu^2$
N_{Gr} = local terrestrial Grashof number, $gx^3\beta\Delta T/\nu^2$
N_{Nu} = Nusselt number, hx/k
N_{Pr} = Prandtl number, $c_p\mu/k$
t = time
T = temperature
u = velocity parallel to plate
U = dimensionless parallel velocity, $u/(\omega^2\nu^2\beta\Delta T)^{1/4}$
v = velocity normal to plate
V = dimensionless normal velocity, $v/(\omega^2\nu^2\beta\Delta T)^{1/4}$
x = distance (parallel to plate) from leading edge
X = dimensionless distance, $x(\omega^2\beta\Delta T/\nu^2)^{1/4}$
y = distance from plate
Y = dimensionless distance, $y(\omega^2\beta\Delta T/\nu^2)^{1/4}$

Greek Letters

β = coefficient of volumetric expansion
ΔT = overall temperature difference, $T_i - T_p$
μ = dynamic viscosity
ν = kinematic viscosity
ρ = density
ϕ = dimensionless temperature, $(T_i - T)/\Delta T$
τ = dimensionless time, $t\omega\sqrt{\beta\Delta T}$
ω = angular velocity

Subscripts

i = initial
m = integer denoting grid position parallel to plate
n = integer denoting grid position normal to plate
p = plate

Superscript

$'$ = after time interval $\Delta\tau$ (except for N'_{Gr})

LITERATURE CITED

1. Crawford, L., and R. Lemlich, *Ind. Eng. Chem. Fundamentals Quart.*, 1, 260-264 (1962).
2. Eckert, E. R. G., and R. M. Drake, Jr., "Heat and Mass Transfer," p. 313, McGraw-Hill, New York (1959).
3. Hellums, J. D., and S. W. Churchill, *Chem. Eng. Progr. Symposium Ser. No. 32*, **57**, 75-80 (1961).
4. ———, "International Development in Heat Transfer," pp. 985-994, Am. Soc. Mech. Engrs., New York (1961).
5. ———, *A.I.Ch.E. Journal*, **8**, 690-695, 719 (1962).
6. Jakob, M., "Heat Transfer," Vol. 1. p. 254, Wiley, New York (1949).
7. Lemlich, R., *Ind. Eng. Chem. Fundamentals Quart.*, **2**, 157-158 (1963).
8. Ostrach, S., *Natl. Advisory Comm. Aeronaut. Rept. 1111* (1953).
9. Schmidt, E., and W. Beckmann, *Tech. Mech. u. Thermodynamik*, **1**, 341-349, 391-406 (1930).
10. Sparrow, E. M., and J. L. Gregg, *Trans. Am. Soc. Mech. Engrs.*, **80**, 379-386 (1958).
11. *Ibid.*, **78**, 435-440 (1956).
12. Steinkamp, J. S., M.S. thesis, University of Cincinnati, Cincinnati, Ohio (1963).

Manuscript received April 17, 1963; revision received December 2, 1963; paper accepted December 2, 1963.